Ehrhart Theory and Graph Colorings

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Outline

1. Classical Ehrhart theory
2. Graph colorings
3. The q-analog connection

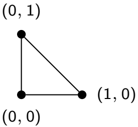
Lattice polytopes

A polytope is the convex hull of finitely many points in $\mathbb{R}^d$, equivalently a bounded intersection of finitely many halfspaces.

For P a lattice polytope (i.e. with vertices in $\mathbb{Z}^d$), we consider

$$\mathrm{ehr}_P(n) = \left| nP \cap \mathbb{Z}^d \right|.$$

Example:

$\Delta =$ 

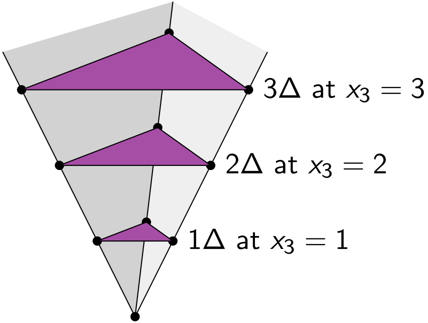
$$\mathrm{ehr}_\Delta(n) = |\{(x,y) \in \mathbb{Z}^2 \,:\, x, y \geq 0, x + y \leq n\}|$$
$$= \binom{n+2}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1$$

Ehrhart polynomials and series

For any d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$, $\mathrm{ehr}_P(n)$ is a polynomial of degree d, called the **Ehrhart polynomial**.

The **Ehrhart series** of P is its generating function

$$\mathrm{Ehr}_P(z) = \sum_{n \geq 0} \mathrm{ehr}_P(n) z^n.$$

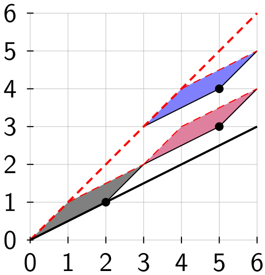


Ehrhart theory of unimodular simplices

If Δ is a d-dimensional *unimodular* simplex with k missing facets (for some $0 \leq k \leq d+1$),

$$\mathrm{Ehr}_\Delta(z) = \frac{z^k}{(1-z)^{d+1}}.$$

cone((1, 2]) :



Ehrhart theory of order polytopes

The **order polytope** of a poset $\Pi = ([d], \preceq)$ is

$$\mathcal{O}(\Pi) = \{(x_1, \ldots, x_d) \in [0,1]^d \,:\, x_i \le x_j \text{ if } i \preceq j\},$$

which has a disjoint unimodular triangulation

$$\mathcal{O}(\Pi) = \bigcup_{\sigma \in \mathcal{L}(\Pi)} \left\{0 \le x_{\sigma_1} \le \ldots \le x_{\sigma_d} \le 1,\, x_{\sigma_i} < x_{\sigma_{i+1}} \text{ if } i \in \mathrm{Des}(\sigma)\right\}.$$

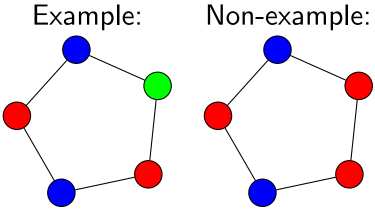
Therefore,

$$\mathrm{Ehr}_{\mathcal{O}(\Pi)}(z) = \frac{\sum_{\sigma \in \mathcal{L}(\Pi)} z^{\mathrm{des}(\sigma)}}{(1-z)^{d+1}}.$$

Proper colorings

A **proper n-coloring** of a graph $G = (V, E)$ is a function $c : V \to [n]$ such that

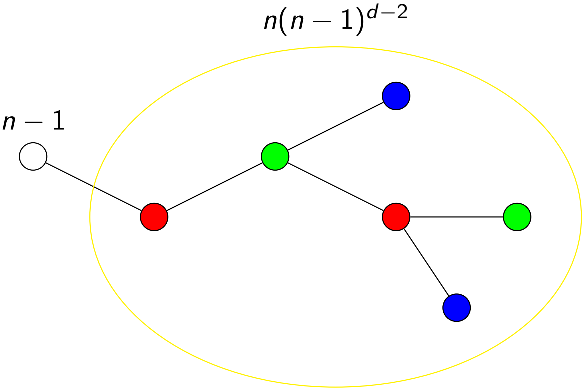
$$c(v) \neq c(w) \text{ if } \{v, w\} \in E.$$



The number of proper n-colorings of a graph G agrees with a polynomial of degree $|V|$, called the **chromatic polynomial** $\chi_G(n)$ of G.

The chromatic polynomial of a tree

If T is a tree on d vertices, then $\chi_T(n) = n(n-1)^{d-1}$.



Proper colorings as lattice points

A coloring $c : [d] \to [n]$ of $G = ([d], E)$ can be thought of as a point

$$(c(1), \ldots, c(d)) \in \mathbb{Z}^d.$$

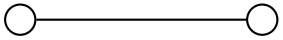
The proper n-colorings of G are points in

$$\left((0, n+1)^d \cap \mathbb{Z}^d\right) \setminus \left(\bigcup \mathcal{H}_G\right),$$

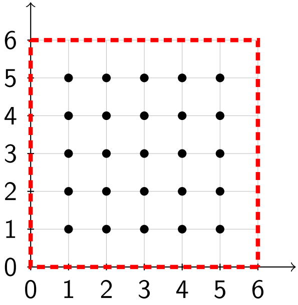
where $\mathcal{H}_G$ is the **graphical hyperplane arrangement**

$$\mathcal{H}_G = \{x_i = x_j \,:\, \{i, j\} \in E\}.$$

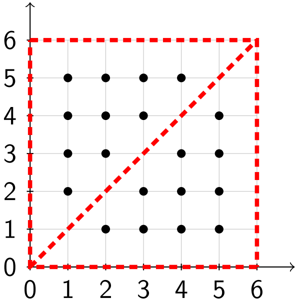
Proper colorings as lattice points, continued

Consider the path on two vertices, $P_2 =$ 

5-colorings of P_2:



Proper 5-colorings of P_2:

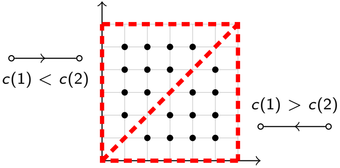


Proper colorings as lattice points, continued

$\left((0,n+1)^d \cap \mathbb{Z}^d\right) \setminus \left(\bigcup \mathcal{H}_G\right)$ has a region for each *acyclic orientation* ρ of G, given by

$$(0,n+1)^d \cap \left(\bigcap_{(i,j)\in\rho} \{x_i < x_j\} \right).$$

The region corresponding to ρ contains the proper colorings of G that "obey" ρ, i.e. for which $c(i) < c(j)$ if $(i,j) \in \rho$.



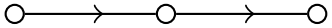
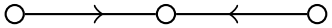
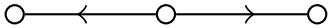
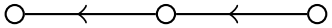
The chromatic polynomial is a sum of Ehrhart polynomials

Each region is the $(n+1)$st dilate of the open order polytope of the poset induced by ρ, which we call Π_ρ, therefore

$$\begin{aligned}\chi_G(n) &= \sum_{\rho \in \mathcal{A}(G)} \mathrm{ehr}_{\mathcal{O}(\Pi_\rho)^\circ}(n+1) \\ &= \sum_{\rho \in \mathcal{A}(G)} \sum_{\sigma \in \mathcal{L}(\Pi_\rho)} \binom{n + \mathrm{des}(\sigma)}{d}.\end{aligned}$$

The linear extensions are of a *natural labeling* of the poset, not the vertex labels.

An example: the path on 3 vertices

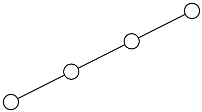
Acyclic Orientation ρ	Induced Poset Π_ρ	Linear Extensions $\mathcal{L}(\Pi_\rho)$
		123
		123, $\underline{2}13$
		123, $1\underline{3}2$
		123

$$\chi_{P_3}(n) = 4\binom{n}{3} + 2\binom{n+1}{3} = n(n-1)^2$$

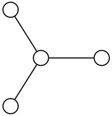
The chromatic symmetric function

Stanley's symmetric function generalization:

$$X_G(x_1, x_2, \ldots) = \sum_{\substack{\text{proper colorings} \\ c: V \to \mathbb{Z}^+}} x_1^{|c^{-1}(1)|} x_2^{|c^{-1}(2)|} x_3^{|c^{-1}(3)|} \ldots$$



$$X_{P_4}(x_1, x_2, 0, 0, \ldots) = 2x_1^2 x_2^2$$



$$X_{S_4}(x_1, x_2, 0, 0, \ldots) = x_1^3 x_2 + x_1 x_2^3$$

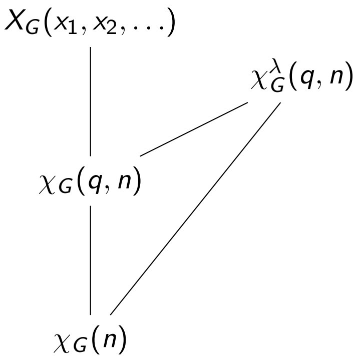
The big picture

Stanley's chromatic symmetric function $X_G(x_1, x_2, \ldots)$:

- Stanley: Distinguishes non-isomorphic trees?
- Loehr-Warrington: So does $X_G(q, q^2, \ldots, q^n, 0, 0 \ldots)$?

Chromatic polynomial $\chi_G(n)$:

- Polytopes perspective
- Deletion-contraction
- Does not distinguish trees



Chapoton's q-analog Ehrhart theory

Theorem. (Chapoton) If $P \subseteq \mathbb{R}^d$ is a d-dimensional lattice polytope and $\lambda : \mathbb{Z}^d \to \mathbb{Z}$ is a linear form that is nonnegative on the vertices of P,

$$\mathrm{ehr}_P^{\lambda}(q,n) = \sum_{x \in nP \cap \mathbb{Z}^d} q^{\lambda(x)}$$

agrees with a polynomial $\widetilde{\mathrm{ehr}}_P^{\lambda}(q,x) \in \mathbb{Q}(q)[x]$, evaluated at

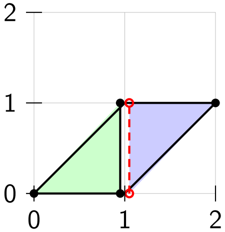
$$x = [n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

If $\lambda((x_1,\ldots,x_d)) = x_1 + \cdots + x_d$, we omit it.

We are ignoring a condition called "genericity" that is needed, but we will not have to worry about it for the polytopes we are working with!

An example of $\widetilde{\text{ehr}}$!

$$P = \text{conv}\{(0,0),(1,0),(1,1),(2,1)\}$$



$$\text{Ehr}_P(q,z) = \frac{1}{(1-z)(1-qz)(1-q^2z)} + \frac{q^3z}{(1-qz)(1-q^2z)(1-q^3z)}$$
$$= \frac{1-q^3z^2}{(1-z)(1-qz)(1-q^2z)(1-q^3z)}$$

$$\widetilde{\text{ehr}}_P(q,x) = \frac{q^4-q^3}{q+1}x^3 + \frac{3q^3-q^2}{q+1}x^2 + \frac{3q^2+q}{q+1}x + 1$$

The weighted connection between Ehrhart theory and graph colorings

$$X_G(q, q^2, \ldots, q^n, 0, \ldots) = \sum_{\substack{\text{proper} \\ c:[d]\to[n]}} q^{|c^{-1}(1)|+2|c^{-1}(2)|+\cdots+n|c^{-1}(n)|}$$

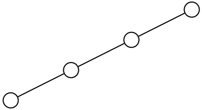
counts q raised to the *sum of the colors of each vertex* for each proper coloring, which is

$$\chi_G(q,n) := \sum_{\rho\in\mathcal{A}(G)} \mathrm{ehr}_{\mathcal{O}(\Pi_\rho)^\circ}(q, n+1).$$

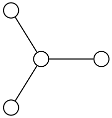
Therefore,

$$X_G(q, q^2, \ldots, q^n, 0, \ldots) = \sum_{\rho\in\mathcal{A}(G)} \sum_{\sigma\in\mathcal{L}(\Pi_\rho)} q^{\binom{d+1}{2}-\mathrm{comaj}(\sigma)} \begin{bmatrix} n+\mathrm{des}(\sigma) \\ d \end{bmatrix}_q$$

Some examples of $\chi_T(q,n)$ in the "h^*-basis"



$$8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (4q^9+6q^8+4q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + 2q^6\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$$



$$8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (5q^9+4q^8+5q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + (q^7+q^5)\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$$

The q-analog chromatic polynomial

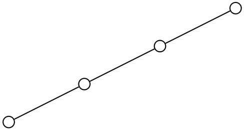
There exists a polynomial $\widetilde{\chi}_G(q,x) \in \mathbb{Q}(q)[x]$, which we call the q-**analog chromatic polynomial**, such that

$$\widetilde{\chi}_G(q,[n]_q) = \chi_G(q,n) \quad \big(= X_G(q,q^2,\ldots,q^n,0,\ldots)\big).$$

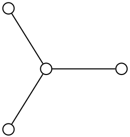
Theorem.

$$\widetilde{\chi}_G(q,x) = q^d \sum_{\text{flats } S \subseteq E} \mu(\varnothing,S) \prod_{\lambda_i \in \lambda(S)} \frac{1-(1+(q-1)x)^{\lambda_i}}{1-q^{\lambda_i}}$$

Some examples of $[d]_q! \cdot \widetilde{\chi}_T(q,x)$



$(2q^8+4q^7+6q^6+4q^5+8q^4)x^4+$
$(-6q^8-10q^7-18q^6-18q^5-20q^4)x^3+$
$(4q^8+10q^7+20q^6+22q^5+16q^4)x^2+$
$(-4q^7-8q^6-8q^5-4q^4)x$



$(q^9+6q^7+4q^6+5q^5+8q^4)x^4+$
$(-q^9-3q^8-14q^7-14q^6-21q^5-19q^4)x^3+$
$(3q^8+12q^7+18q^6+24q^5+15q^4)x^2+$
$(-4q^7-8q^6-8q^5-4q^4)x$

Conjecture. The *leading coefficient* distinguishes trees.

3 ways to compute: h^*-basis, Möbius inversion, deletion-contraction

The q, λ-analog chromatic polynomial

Chapoton's weighted Ehrhart theory applies to general linear forms λ, so we can also define:

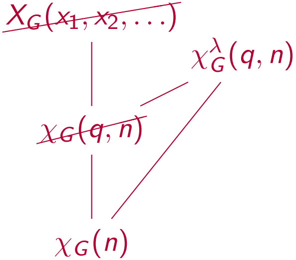
$$\chi_G^\lambda(q,n) := \sum_{\substack{\text{proper} \\ c:[d]\to[n]}} q^{\lambda_1 c(1)+\cdots+\lambda_d c(d)}$$
$$= \sum_{\rho\in\mathcal{A}(G)} \mathrm{ehr}^\lambda_{\mathcal{O}(\Pi_\rho)^\circ}(q, n+1).$$

The bad news: For general λ, χ_G^λ is not a necessarily an instance of the chromatic symmetric function.

Why care about χ_G^λ (and $\widetilde{\chi}_G^\lambda$)?

Deletion-Contraction Lemma. Let $G = ([d], E)$ be a graph with $e = \{1, 2\} \in E$. Then

$$\chi_G^{(\lambda_1,\ldots,\lambda_d)}(q,n) = \chi_{G\setminus e}^{(\lambda_1,\ldots,\lambda_d)}(q,n) - \chi_{G/e}^{(\lambda_1+\lambda_2,\ldots,\lambda_n)}(q,n).$$



Conjecture. If S and T are non-isomorphic trees, then there exists λ for which

$$\chi_S^\lambda(q,n) \neq \chi_T^\lambda(q,n).$$

1. F. Chapoton. q-analogues of Ehrhart polynomials. *Proc. Edinb. Math. Soc.*, (2) 59 (2016), no. 2, 339–358.
2. R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111(1):166–194, 1995.
3. N. A. Loehr and G. S. Warrington. A rooted variant of Stanley's chromatic symmetric function. (arXiv:2206.05392)

Thank you!! :)