Weighted Ehrhart Theories

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Outline

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4. The second weighting
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   - Combinatorial connections

Permutation statistics

For $\pi = \pi_1 \pi_2 \ldots \pi_d$ a permutation: $\underline{3}\,1\,5\,\underline{7}\,\underline{4}\,2\,6$

- $\mathrm{Des}(\pi) := \{i \in [d-1] : \pi_i > \pi_{i+1}\}$ $\{1,4,5\}$
- $\mathrm{des}(\pi) := |\mathrm{Des}(\pi)|$ 3
- $\mathrm{maj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} i$ $1+4+5=10$
- $\mathrm{comaj}(\pi) := \sum_{i \in \mathrm{Des}(\pi)} (d-i)$ $6+3+2=11$

Eulerian polynomials

The dth Eulerian polynomial is $A_d(z) := \sum_{\pi \in S_d} z^{\mathrm{des}(\pi)}$.

$1\,2\,3 \quad 1\,\underline{3}\,2 \quad \underline{3}\,\underline{2}\,1$

$\underline{2}\,1\,3$

$2\,\underline{3}\,1 \qquad A_3(z) = 1 + 4z + z^2$

$\underline{3}\,1\,2$

A generating function involving Eulerian polynomials:

$$\sum_{n \geq 0} (n+1)^d z^n = \frac{A_d(z)}{(1-z)^{d+1}}$$

Generalized Eulerian polynomials

Rational expressions of generating functions: A sequence $f(n)$ is given by a polynomial of degree $\leq d$ if and only if

$$\sum_{n\geq 0} f(n)z^n = \frac{h(z)}{(1-z)^{d+1}}$$

for some polynomial $h(z)$ of degree $\leq d$.

For $\lambda \in [0,1]$, let $A_n^\lambda(z)$ be the polynomial defined by

$$\sum_{n\geq 0}(n+\lambda)^d z^n = \frac{A_d^\lambda(z)}{(1-z)^{d+1}}.$$

$A_d^\lambda(z)$ has *nonnegative coefficients*.

q-binomial coefficients

The q-integer:

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$$

The q-binomial coefficient:

$$\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q := \frac{[k+\ell]_q!}{[k]_q![\ell]_q!} = \frac{[k+\ell]_q[k+\ell-1]_q \cdots [k+1]_q}{[\ell]_q[\ell-1]_q \cdots [1]_q}$$

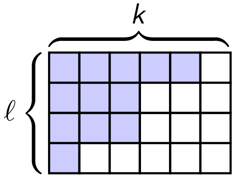
q-analog of Pascal's identity:

$$\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = q^k \begin{bmatrix} k+(\ell-1) \\ k \end{bmatrix}_q + \begin{bmatrix} (k-1)+\ell \\ k-1 \end{bmatrix}_q$$

q-binomial coefficients, continued

A combinatorial interpretation:

$$\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \sum_{\mu \in \mathcal{R}(k,\ell)} q^{|\mu|}$$



Negative q-binomial theorem:

$$\frac{1}{(1-z)(1-qz)(1-q^2z)\cdots(1-q^dz)} = \sum_{n \geq 0} \begin{bmatrix} n+d \\ d \end{bmatrix}_q z^n$$

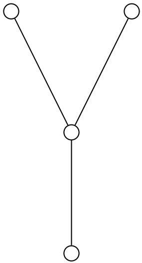
Posets

$\Pi = (P, \preceq)$ such that for all $p, q, r \in P$:

- $p \preceq p$
- $p \preceq q$ and $q \preceq p \implies p = q$
- $p \preceq q$ and $q \preceq r \implies p \preceq r$

q **covers** p if $p \prec q$ and if there is no r such that $p \prec r \prec q$.

Hasse diagram:



Linear extensions and natural labelings

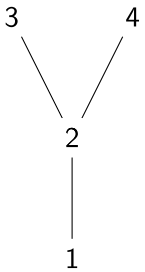
Fix a labeling of Π, i.e. a bijection $\omega : P \to [n]$. The **linear extensions** of the labeled poset are the order-preserving maps

$$\mathcal{L}(\Pi) := \{\sigma \in S_n : \sigma(\omega(p)) < \sigma(\omega(q)) \text{ if } p \prec q\}.$$

A labeling is **natural** if the identity is a linear extension.

An example!

Natural labeling:



Linear extensions:

$$\mathcal{L}(\Pi) = \{1234, 1243\}$$

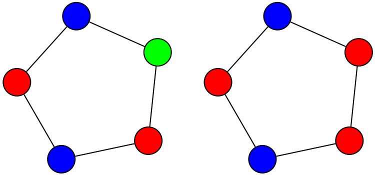
Proper colorings

A **proper** n**-coloring** of a graph $G = (V, E)$ is a function $c : V \to [n]$ such that

$$c(v) \neq c(w) \text{ if } \{v, w\} \in E.$$

The **chromatic number** $\chi(G)$ of G is the smallest positive integer such that G has a proper $\chi(G)$-coloring.

Example: Non-example:



$$\chi(C_5) = 3$$

The chromatic polynomial

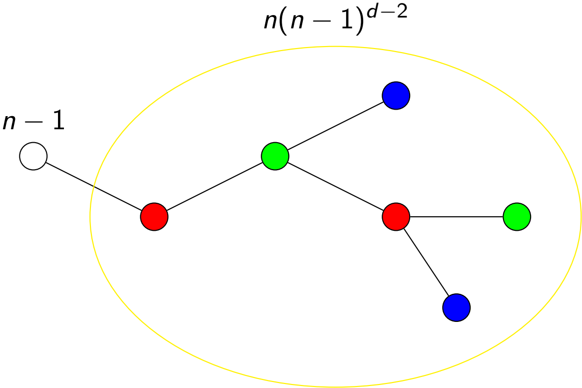
The number of proper n-colorings of a graph G agrees with a polynomial of degree $|V|$, called the **chromatic polynomial** $\chi_G(n)$ of G.

$$\chi_G(n) = \sum_{k=\chi(G)}^{|V|} \alpha_k \cdot n(n-1)\cdots(n-k+1),$$

where α_k is the number of partitions of V into k independent sets.

The chromatic polynomial of a tree

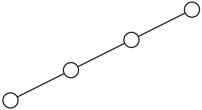
If T is a tree on d vertices, then $\chi_T(n) = n(n-1)^{d-1}$.



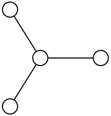
The chromatic symmetric function

Stanley's symmetric function generalization:

$$X_G(x_1, x_2, \ldots) = \sum_{\substack{\text{proper colorings} \\ c: V \to \mathbb{Z}^+}} x_1^{|c^{-1}(1)|} x_2^{|c^{-1}(2)|} x_3^{|c^{-1}(3)|} \ldots$$



$$X_{P_4}(x_1, x_2, 0, 0, \ldots) = 2x_1^2 x_2^2$$



$$X_{S_4}(x_1, x_2, 0, 0, \ldots) = x_1^3 x_2 + x_1 x_2^3$$

The chromatic symmetric function in different bases

(Augmented) monomial basis

$$X_G(x_1, x_2, \ldots) = \sum_{\lambda \vdash |V|} \alpha_\lambda \widetilde{m}_\lambda,$$

where α_λ = number of partitions of type λ of V into independent sets and $\widetilde{m}_\lambda = r_1! r_2! \cdots m_\lambda$ (r_i = number of parts of λ equal to i)

Power sum basis

$$X_G(x_1, x_2, \ldots) = \sum_{S \subseteq E} (-1)^{|S|} p_{\lambda(S)},$$

where $\lambda(S)$ = vector of sizes of connected components of (V, S)

Elementary basis

$$X_G(x_1, x_2, \ldots) = \sum_{\lambda \vdash |V|} c_\lambda e_\lambda,$$

is such that $\sum_{\substack{\lambda \text{ with} \\ j \text{ parts}}} c_\lambda$ = number of acyclic orientations of G with j sinks

Conjectures about $X_G(x_1, x_2, \ldots)$

1. [Stanley] $\boxed{\text{For trees } S \text{ and } T,\ X_S = X_T \iff S \cong T.}$
2. [Stanley] Chromatic symmetric functions of claw-free graphs are Schur positive.
3. [Stanley-Stembridge] Chromatic symmetric functions of incomparability graphs of $(3+1)$-free posets are e-positive.

Specializations of $X_G(x_1, x_2, \ldots)$

$$X_G(x_1, x_2, \ldots)$$

$$X_G(q, q^2, \ldots, q^n, 0, 0, \ldots)$$

Conjecture. (Loehr-Warrington) The **principal specialization** already distinguishes non-isomorphic trees!

$$X_G(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots) = \chi_G(n)$$

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Lattice polytopes

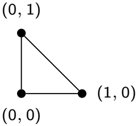
A polytope is the convex hull of finitely many points in $\mathbb{R}^d$, equivalently a bounded intersection of finitely many halfspaces.

For P a lattice polytope (i.e. with vertices in $\mathbb{Z}^d$), we consider

$$\mathrm{ehr}_P(n) = \left| nP \cap \mathbb{Z}^d \right| .$$

Example:

$\Delta =$



$$\begin{aligned}\mathrm{ehr}_\Delta(n) &= |\{(x,y) \in \mathbb{Z}^2 \,:\, x, y \geq 0, x + y \leq n\}| \\ &= \binom{n+2}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1\end{aligned}$$

Ehrhart polynomials and series

For any d-dimensional lattice polytope $P \subseteq \mathbb{R}^d$, $\mathrm{ehr}_P(n)$ is a polynomial of degree d, called the **Ehrhart polynomial**.

The **Ehrhart series** of P is its generating function

$$\mathrm{Ehr}_P(z) = \sum_{n \geq 0} \mathrm{ehr}_P(n) z^n.$$

Observe

$$\mathrm{Ehr}_P(z) = \sum_{x \in \mathrm{cone}(P) \cap \mathbb{Z}^{d+1}} z^{x_{d+1}},$$

where $\mathrm{cone}(P) = \{(tx, t) : x \in P, t \geq 0\}$.

Ehrhart theory of unimodular simplices

If Δ is a d-dimensional *unimodular* simplex with k missing facets (for some $0 \le k \le d+1$),

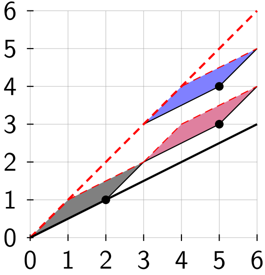
$$\mathrm{Ehr}_\Delta(z) = \frac{z^k}{(1-z)^{d+1}}.$$

Proof. The unique point in the "fundamental parallelepiped" of cone(Δ) is

$$\sum \binom{v_i}{i},$$

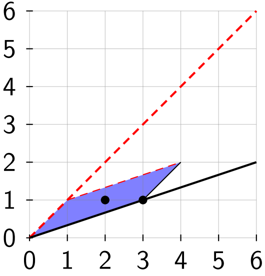
where the sum ranges over the k vertices of Δ that are opposite the missing facets.

cone$((1,2])$:



Ehrhart theory of general lattice simplices

$\mathrm{cone}((1,3])$:



$$\mathrm{Ehr}_\Delta(z) = \frac{\sum_{x\in\Pi(\Delta)\cap\mathbb{Z}^{d+1}} z^{x_{d+1}}}{(1-z)^{d+1}}$$

$$= \frac{2z}{(1-z)^2}$$

h^*-polynomials of lattice polytopes

1. [Nonnegativity] If P is a d-dimensional lattice polytope,

$$\mathrm{Ehr}_P(z) = \frac{h_P^*(z)}{(1-z)^{d+1}},$$

   where $h_P^*(z)$ is a polynomial with nonnegative integer coefficients, called the h^*-**polynomial**.

2. [Monotonicity] If P, Q are lattice polytopes and $P \subseteq Q$,

$$h_P^*(z) \leq h_Q^*(z),$$

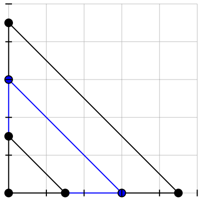
   coefficient-wise.

Rational polytopes and Ehrhart quasipolynomials

If $P \subseteq \mathbb{R}^d$ has rational vertices, say in $\frac{1}{q}\mathbb{Z}^d$ for $q \geq 1$ minimal,

$$|nP \cap \mathbb{Z}^d|$$

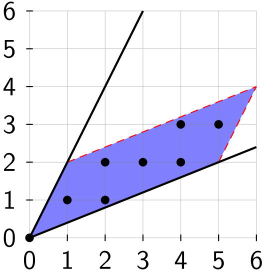
agrees with a **quasipolynomial** in n whose period divides q.



$$|nP \cap \mathbb{Z}^d| = \begin{cases} \frac{9}{8}n^2 + \frac{9}{4}n + 1 & \text{if } n \equiv 0 \bmod 2 \\ \frac{9}{8}n^2 + \frac{3}{2}n + \frac{3}{8} & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

Ehrhart series of rational simplices

$\mathrm{cone}([1/2, 5/2])$:



$$\mathrm{Ehr}_\Delta(z) = \frac{\sum_{x \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} z^{x_{d+1}}}{(1 - z^q)^{d+1}}$$

$$= \frac{1 + 2z + 3z^2 + 2z^3}{(1 - z^2)^2}$$

h^*-polynomials of rational polytopes

1. [Nonnegativity] If P is a d-dimensional rational polytope with denominator q,

$$\mathrm{Ehr}_P(z) = \frac{h_P^*(z)}{(1-z^q)^{d+1}},$$

where $h_P^*(z)$ is a polynomial with nonnegative integer coefficients, called the h^*-**polynomial**.

2. [Monotonicity] If P, Q are rational polytopes of the same denominator and $P \subseteq Q$,

$$h_P^*(z) \leq h_Q^*(z),$$

coefficient-wise.

Unit cubes

The d-dimensional unit cube has a disjoint *unimodular triangulation*

$$[0,1]^d = \bigcup_{\sigma \in S_d} \{0 \leq x_{\sigma_1} \leq \cdots \leq x_{\sigma_d} \leq 1 : x_{\sigma_i} < x_{\sigma_{i+1}} \text{ if } i \in \mathrm{Des}(\sigma)\},$$

so

$$\mathrm{Ehr}_{[0,1]^d}(z) = \frac{\sum_{\sigma \in S_d} z^{\mathrm{des}(\sigma)}}{(1-z)^{d+1}}$$

$$\implies \sum_{n \geq 0} (n+1)^d z^n = \frac{A_d(z)}{(1-z)^{d+1}}$$

Order polytopes

The **order polytope** of a poset $\Pi = ([d], \preceq)$ is

$$\mathcal{O}(\Pi) = \{(x_1, \ldots, x_d) \in [0,1]^d \,:\, x_i \leq x_j \text{ if } i \preceq j\},$$

which has a disjoint unimodular triangulation

$$\mathcal{O}(\Pi) = \bigcup_{\sigma \in \mathcal{L}(\Pi)} \left\{0 \leq x_{\sigma_1} \leq \ldots \leq x_{\sigma_d} \leq 1,\, x_{\sigma_i} < x_{\sigma_{i+1}} \text{ if } i \in \mathrm{Des}(\sigma)\right\}.$$

Therefore,

$$\mathrm{Ehr}_{\mathcal{O}(\Pi)}(z) = \frac{\sum_{\sigma \in \mathcal{L}(\Pi)} z^{\mathrm{des}(\sigma)}}{(1-z)^{d+1}}.$$

Order polytopes, continued

The Negative Binomial Theorem implies

$$\mathrm{ehr}_{\mathcal{O}(\Pi)}(n) = \sum_{\sigma \in \mathcal{L}(\Pi)} \binom{n + d - \mathrm{des}(\sigma)}{d}$$

and **Ehrhart-Macdonald reciprocity** implies

$$\mathrm{ehr}_{\mathcal{O}(\Pi)^\circ}(n) = \sum_{\sigma \in \mathcal{L}(\Pi)} \binom{n + \mathrm{des}(\sigma) - 1}{d}$$

Proper colorings as lattice points

A coloring $c : [d] \to [n]$ of $G = ([d], E)$ can be thought of as a point

$$(c(1), \ldots, c(d)) \in \mathbb{Z}^d.$$

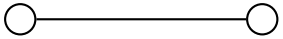
The proper n-colorings of G are points in

$$\left((0, n+1)^d \cap \mathbb{Z}^d\right) \setminus \left(\bigcup \mathcal{H}_G\right),$$

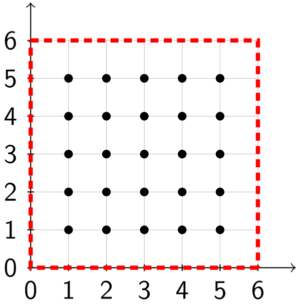
where $\mathcal{H}_G$ is the **graphical hyperplane arrangement**

$$\mathcal{H}_G = \{x_i = x_j \, : \, \{i, j\} \in E\}.$$

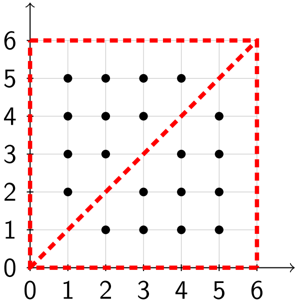
Proper colorings as lattice points, continued

Consider the path on two vertices, $P_2 = $ 

5-colorings of P_2:



Proper 5-colorings of P_2:

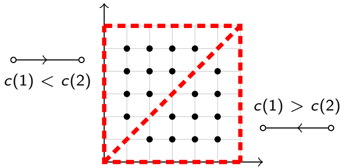


Proper colorings as lattice points, continued

$\left((0, n+1)^d \cap \mathbb{Z}^d\right) \setminus \left(\bigcup \mathcal{H}_G\right)$ has a region for each *acyclic orientation* ρ of G, given by

$$(0, n+1)^d \cap \left(\bigcap_{(i,j)\in\rho} \{x_i < x_j\} \right).$$

The region corresponding to ρ contains the proper colorings of G that "obey" ρ, i.e. for which $c(i) < c(j)$ if $(i, j) \in \rho$.



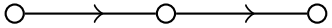
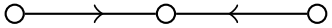
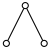
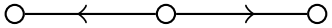
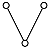
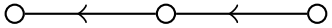
The chromatic polynomial is a sum of Ehrhart polynomials

Each region is the $(n+1)$st dilate of the open order polytope of the poset induced by ρ, which we call Π_ρ, therefore

$$\begin{aligned}\chi_G(n) &= \sum_{\rho \in \mathcal{A}(G)} \mathrm{ehr}_{\mathcal{O}(\Pi_\rho)^\circ}(n+1) \\ &= \sum_{\rho \in \mathcal{A}(G)} \sum_{\sigma \in \mathcal{L}(\Pi_\rho)} \binom{n+\mathrm{des}(\sigma)}{d}.\end{aligned}$$

The linear extensions are of a *natural labeling* of the poset, not the vertex labels.

An example: the path on 3 vertices

Acyclic Orientation ρ	Induced Poset Π_ρ	Linear Extensions $\mathcal{L}(\Pi_\rho)$
		123
		123, $\underline{2}13$
		123, $1\underline{3}2$
		123

$$\chi_{P_3}(n) = 4\binom{n}{3} + 2\binom{n+1}{3} = n(n-1)^2$$

Leading questions

1. What kind of “weights” can we introduce to the lattice so that classical Ehrhart results will generalize?
2. Will meaningful combinatorial connections (to posets, graphs, etc.) arise in the weighted versions?

The first weighting

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Sofía Garzón Mora

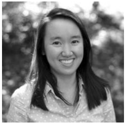
Freie Universität Berlin

Katharina Jochemko

KTH Royal Institute of Technology

Josephine Yu

Georgia Institute of Technology



Lots of "weighted Ehrhart theories" have been studied!

$$\mathrm{ehr}_P(\omega, n) = \sum_{x \in nP \cap \mathbb{Z}^d} \omega(x)$$

- [Stapledon '08] piecewise linear functions
- [Chapoton '16] "q-analog" Ehrhart theory, $\omega(x) = q^{\lambda(x)}$
- [Ludwig-Silverstein '17] tensor valuations

Our setup

Let $\omega : \mathbb{R}^d \to \mathbb{R}$ be a polynomial of degree m and let $P \subseteq \mathbb{R}^d$ be a d-dimensional rational polytope with denominator q. The **weighted Ehrhart series**

$$\mathrm{Ehr}(P, \omega; z) := \sum_{n \geq 0} \left(\sum_{x \in nP \cap \mathbb{Z}^d} \omega(x) \right) z^n$$

is a rational function of the form

$$\mathrm{Ehr}(P, \omega; z) = \frac{h^*_{P,\omega}(z)}{(1 - z^q)^{d+m+1}},$$

where $h^*_{P,\omega}(z)$ is a polynomial of degree $< q(d + m + 1)$.

What changes?

The weighted h^*-polynomial does not have to have nonnegative coefficients anymore!

Example: For $P = [0,1]$,

$$\mathrm{Ehr}(P,1;z) = \frac{1}{(1-z)^2} \text{ and } \mathrm{Ehr}(P,x^2;z) = \frac{z^2+z}{(1-z)^4},$$

$$\text{so } \mathrm{Ehr}(P,x^2+1;z) = \frac{2z^2-z+1}{(1-z)^4}.$$

For this reason (not introducing negatives while getting a LCD) we will focus on **homogeneous** weight polynomials. But this is not enough – negative coefficients will still pop up!

When ω is a product of linear forms...

Lemma. If $\Delta = \operatorname{conv}\{v_1, \ldots, v_{d+1}\} \subseteq \mathbb{R}^d$ is a d-dimensional half-open rational simplex with denominator q and ω is a product of m linear forms $\ell_1 \cdots \ell_m$,

$$h^*_{\Delta,w}(z) = \sum_{x \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} \left(z^{x_{d+1}} \sum_{I_1 \uplus \ldots \uplus I_{d+1} = [m]} \prod_{i \in I_1} \ell_i(v_1) \cdots \prod_{i \in I_{d+1}} \ell_i(v_{d+1}) \prod_{j=1}^{r+1} A_{|I_j|}^{\lambda_j(x)}(z^q) \right)$$

where $x = \lambda_1(x) \begin{pmatrix} qv_1 \\ q \end{pmatrix} + \cdots + \lambda_{d+1}(x) \begin{pmatrix} qv_{d+1} \\ q \end{pmatrix}$.

Positive consequences!

Theorem 1 (Nonnegativity). If ω is a homogeneous sum of products of linear forms that are nonnegative on the rational polytope P, then $h^*_{P,\omega}(z)$ has nonnegative coefficients.

Positive consequences, continued

Theorem 2 (Monotonicity). Let $P \subseteq Q$ be rational polytopes with denominators $\delta(P)$ and $\delta(Q)$, respectively. If g is any common multiple of $\delta(P)$ and $\delta(Q)$ and ω is a homogeneous (degree m) sum of products of linear forms that are nonnegative on Q, then

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{\dim(P)+m+1} h^*_{P,\omega}(z) \leq$$
$$(1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{\dim(Q)+m+1} h^*_{Q,\omega}(z),$$

coefficient-wise.

Do we really need these assumptions?

ω being nonnegative on P (rather than requiring that each ℓ_i be nonnegative) is not enough!

- $P = \mathrm{conv}\{(0,0),(1,0),(0,1)\}$
- $\omega(x) = (2x_1 - x_2)^2(2x_2 - x_1)^2$
- $h^*_{P,\omega}(z) = z^4 - 6z^3 + 81z^2 + 8z$

There is also a 20-dimensional counterexample for ω just the square of a linear form.

The second weighting

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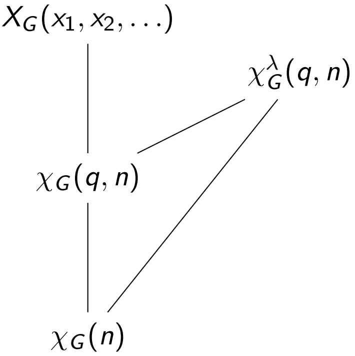
The big picture

Stanley's chromatic symmetric function $X_G(x_1, x_2, \ldots)$:

- Distinguishes some (all?) non-isomorphic trees

Chromatic polynomial $\chi_G(n)$:

- Polytopes perspective
- Deletion-contraction
- Does not distinguish trees



q-analog Ehrhart theory

Theorem. (Chapoton) If $P \subseteq \mathbb{R}^d$ is a d-dimensional lattice polytope and $\lambda : \mathbb{Z}^d \to \mathbb{Z}$ is a linear form that is nonnegative on the vertices of P,

$$\mathrm{ehr}_P^\lambda(q,n) = \sum_{x \in nP \cap \mathbb{Z}^d} q^{\lambda(x)}$$

agrees with a polynomial $\widetilde{\mathrm{ehr}}_P^\lambda(q,x) \in \mathbb{Q}(q)[x]$, evaluated at

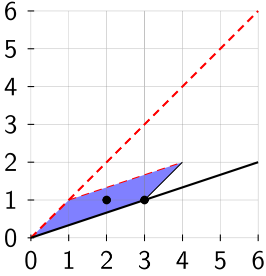
$$x = [n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

If $\lambda((x_1,\ldots,x_d)) = x_1 + \cdots + x_d$, we omit it.

We are ignoring a condition called "genericity" that is needed, but we will not have to worry about it for the polytopes we are working with!

q-analog Ehrhart series of lattice simplices

cone((1, 3]) :

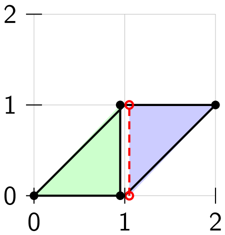


$$\mathrm{Ehr}_{\Delta}^{\lambda}(q,z) = \frac{\sum_{x\in\Pi(\Delta)\cap\mathbb{Z}^{d+1}} q^{\lambda((x_1,\dots,x_d))} z^{x_{d+1}}}{\prod_v (1-q^{\lambda(v)}z)}$$

$$= \frac{q^2z+q^3z}{(1-qz)(1-q^3z)}$$

An example of $\widetilde{\mathrm{ehr}}$!

$$P = \mathrm{conv}\{(0,0),(1,0),(1,1),(2,1)\}$$



$$\begin{aligned}\mathrm{Ehr}_P(q,z) &= \frac{1}{(1-z)(1-qz)(1-q^2z)} + \frac{q^3z}{(1-qz)(1-q^2z)(1-q^3z)}\\ &= \frac{1-q^3z^2}{(1-z)(1-qz)(1-q^2z)(1-q^3z)}\end{aligned}$$

$$\widetilde{\mathrm{ehr}}_P(q,x) = \frac{q^4-q^3}{q+1}x^3 + \frac{3q^3-q^2}{q+1}x^2 + \frac{3q^2+q}{q+1}x + 1$$

Properties of $\mathrm{Ehr}_P^{\lambda}(q,z)$ and $\widetilde{\mathrm{ehr}}_P^{\lambda}(q,x)$

(i) $\mathrm{Ehr}_P^{\lambda}(1,z) = \mathrm{Ehr}_P(z)$ and $\widetilde{\mathrm{ehr}}_P^{\lambda}(1,x) = \mathrm{ehr}_P(x)$

(ii) The denominator of $\mathrm{Ehr}_P^{\lambda}(q,z)$ divides $\displaystyle\prod_{\substack{\text{vertices}\\ v \text{ of } P}} (1 - q^{\lambda(v)}z)$.

(iii) $\deg(\widetilde{\mathrm{ehr}}_P^{\lambda}(q,x)) = \max_v \lambda(v)$

(iv) The poles of the coefficients of $\widetilde{\mathrm{ehr}}_P^{\lambda}(q,x)$ are roots of unity of order at most $\max_v \lambda(v)$.

q-analog Ehrhart theory of unit cubes

Using the same triangulation of the d-dimensional unit cube

$$[0,1]^d = \bigcup_{\sigma \in S_d} \{0 \leq x_{\sigma_1} \leq \cdots \leq x_{\sigma_d} \leq 1 : x_{\sigma_i} < x_{\sigma\, i+1} \text{ if } i \in \mathrm{Des}(\sigma)\},$$

we compute its q-analog Ehrhart series

$$\mathrm{Ehr}_{[0,1]^d}(q,z) = \frac{\sum_{\sigma \in S_d} q^{\mathrm{comaj}(\sigma)} z^{\mathrm{des}(\sigma)}}{(1-z)(1-qz)\cdots(1-q^d z)}.$$

This yields the **Euler-Mahonian joint distribution** of $(\mathrm{des}, \mathrm{maj})$:

$$\sum_{n \geq 0} [n+1]_q^d z^n = \frac{\sum_{\sigma \in S_d} q^{\mathrm{maj}(\sigma)} z^{\mathrm{des}(\sigma)}}{(1-z)(1-qz)\cdots(1-q^d z)}.$$

q-analog Ehrhart theory of order polytopes

The q-analog Ehrhart series of the order polytope $\mathcal{O}(\Pi)$ is

$$\mathrm{Ehr}_{\mathcal{O}(\Pi)}(q,z) = \frac{\sum_{\sigma\in\mathcal{L}(\Pi)} q^{\mathrm{comaj}(\sigma)} z^{\mathrm{des}(\sigma)}}{(1-z)(1-qz)\cdots(1-q^d z)}.$$

Therefore,

$$\mathrm{ehr}_{\mathcal{O}(\Pi)}(q,n) = \sum_{\sigma\in\mathcal{L}(\Pi)} q^{\mathrm{comaj}(\sigma)} \begin{bmatrix} n+d-\mathrm{des}(\sigma) \\ d \end{bmatrix}_q .$$

Observe $[n+k]_q = q^k[n]_q + [k]_q$ and $[n-k]_q = \frac{[n]_q-[k]_q}{q^k}$, so $\widetilde{\mathrm{ehr}}_{\mathcal{O}(\Pi)}(q,x)$ has degree d and $[d]_q! \cdot \widetilde{\mathrm{ehr}}_{\mathcal{O}(\Pi)}(q,x) \in \mathbb{Z}(q)[x]$.

A q-analog connection to graph colorings

$$X_G(q, q^2, \ldots, q^n, 0, \ldots) = \sum_{\substack{\text{proper} \\ c:[d]\to[n]}} q^{|c^{-1}(1)|+2|c^{-1}(2)|+\cdots+n|c^{-1}(n)|}$$

counts q raised to the *sum of the colors of each vertex* for each proper coloring, which is

$$\chi_G(q, n) := \sum_{\rho\in\mathcal{A}(G)} \mathrm{ehr}_{\mathcal{O}(\Pi_\rho)^\circ}(q, n+1).$$

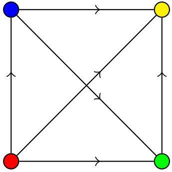
Therefore,

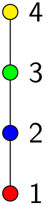
$$X_G(q, q^2, \ldots, q^n, 0, \ldots) = \sum_{\rho\in\mathcal{A}(G)} \sum_{\sigma\in\mathcal{L}(\Pi_\rho)} q^{\binom{d+1}{2}-\mathrm{comaj}(\sigma)} \begin{bmatrix} n+\mathrm{des}(\sigma) \\ d \end{bmatrix}_q$$

A (sort of boring) example

The acyclic orientations of the complete graph K_d are the total orderings of the vertices, which each have the chain as their induced poset.

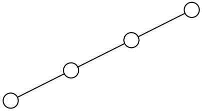
$$\chi_{K_d}(q,n) = d! \cdot q^{\binom{d+1}{2}} \begin{bmatrix} n \\ d \end{bmatrix}_q$$

$\rho =$ 

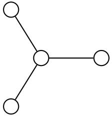
$\Pi_\rho =$ 

$\mathcal{L}(\Pi_\rho) = \{1234\}$

Some examples of $\chi_T(q,n)$ in the "h^*-basis"



$$8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (4q^9+6q^8+4q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + 2q^6\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$$



$$8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (5q^9+4q^8+5q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + (q^7+q^5)\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q$$

The q-analog chromatic polynomial

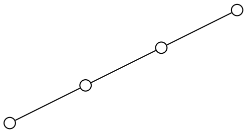
There exists a polynomial $\widetilde{\chi}_G(q,x) \in \mathbb{Q}(q)[x]$, which we call the q-**analog chromatic polynomial**, such that

$$\widetilde{\chi}_G(q,[n]_q) = \chi_G(q,n) \quad \big(= X_G(q,q^2,,q^n,0,\ldots)\big).$$

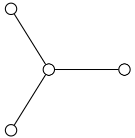
Theorem.

$$\widetilde{\chi}_G(q,x) = q^d \sum_{\text{flats } S \subseteq E} \mu(\varnothing,S) \prod_{\lambda_i \in \lambda(S)} \frac{1-(1+(q-1)x)^{\lambda_i}}{1-q^{\lambda_i}}$$

Some examples of $[d]_q! \cdot \widetilde{\chi}_T(q,x)$



$$(2q^8+4q^7+6q^6+4q^5+8q^4)x^4+$$
$$(-6q^8-10q^7-18q^6-18q^5-20q^4)x^3+$$
$$(4q^8+10q^7+20q^6+22q^5+16q^4)x^2+$$
$$(-4q^7-8q^6-8q^5-4q^4)x$$



$$(q^9+6q^7+4q^6+5q^5+8q^4)x^4+$$
$$(-q^9-3q^8-14q^7-14q^6-21q^5-19q^4)x^3+$$
$$(3q^8+12q^7+18q^6+24q^5+15q^4)x^2+$$
$$(-4q^7-8q^6-8q^5-4q^4)x$$

Conjecture. The *leading coefficient* distinguishes non-isomorphic trees.

The leading coefficient

Theorem. The leading coefficient of $[d]_q! \cdot \widetilde{\chi}_G(q,x)$ is

$$\sum_{\rho \in \mathcal{A}(G)} \sum_{\sigma \in \mathcal{L}(\Pi_\rho)} q^{\mathrm{maj}(\sigma)}.$$

For certain "tree posets" Π and permutation statistics stat,

$$e_q^{\mathrm{stat}}(\Pi) = \sum_{\sigma \in \mathcal{L}(\Pi)} q^{\mathrm{stat}(\sigma)}$$

is well-studied:

- [Björner-Wachs] rooted tree posets, inv
- [Stanley] ribbon posets, inv
- [Peterson-Proctor] d-complete posets, maj
- [Garver-Grosser-Matherne-Morales, Park] mobile tree posets, maj and inv

Open questions!

1. Can these results on "q-analog number of linear extensions" of various tree posets be applied to distinguish the leading coefficients for certain classes of trees?
2. Atkinson gave an algorithm to efficiently compute the number of linear extensions of a tree poset, and Garver-Grosser-Matherne-Morales generalized it for e_q^{inv}. Is there are a major index analog?
3. Generalizing properties of χ to $\widetilde{\chi}$?
   (i) degree d, monic, no constant term
   (ii) integer coefficients, alternating in sign
   (iii) second coefficient is the number of edges
   (iv) linear coefficient is the number of acyclic orientations with a unique sink at some fixed vertex

A reciprocity result

Theorem.

$$(-q)^d \cdot \widetilde{\chi}_G(1/q, -q[n]_q) = \sum_{(\rho,c)} q^{\sum c(i)},$$

where the sum ranges over all pairs of acyclic orientations ρ and *weakly* compatible colorings c (i.e. $c(i) \leq c(j)$ if $(i,j) \in \rho$).

Famous Case: $(-1)^d \cdot \chi_G(-1) = |\mathcal{A}(G)|$

The q, λ-analog chromatic polynomial

Chapoton's weighted Ehrhart theory applies to general linear forms λ, so we can also define:

$$\chi_G^{\lambda}(q,n) := \sum_{\substack{\text{proper} \\ c:[d]\to[n]}} q^{\lambda_1 c(1)+\cdots+\lambda_d c(d)}$$
$$= \sum_{\rho\in\mathcal{A}(G)} \mathrm{ehr}^{\lambda}_{\mathcal{O}(\Pi_\rho)^\circ}(q, n+1).$$

The bad news: For general λ, χ_G^{λ} is not a necessarily an instance of the chromatic symmetric function.

Why care about χ_G^λ (and $\widetilde{\chi}_G^\lambda$)?

Deletion-Contraction Lemma. Let $G = ([d], E)$ be a graph with $e = \{1, 2\} \in E$. Then

$$\chi_G^{(\lambda_1,\ldots,\lambda_d)}(q,n) = \chi_{G\setminus e}^{(\lambda_1,\ldots,\lambda_d)}(q,n) - \chi_{G/e}^{(\lambda_1+\lambda_2,\ldots,\lambda_n)}(q,n).$$

$\cancel{X_G(x_1, x_2, \ldots)}$

$\chi_G^\lambda(q,n)$

$\cancel{\chi_G(q,n)}$

$\chi_G(n)$

Conjecture. If S and T are non-isomorphic trees, then there exists λ for which

$$\chi_S^\lambda(q,n) \neq \chi_T^\lambda(q,n).$$

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Thank you!! :)